

Some generalizations and unifications of $C_K(X)$, $C_\psi(X)$ and $C_\infty(X)$

A. Taherifar

Department of Mathematics, Yasouj University, Yasouj , Iran

ataherifar@mail.yu.ac.ir

Dedicated to professor Azarpanah

Abstract. Let \mathcal{P} be a filter base for \mathcal{F} on X , in which any element is an open subset. We denote by $C_{\mathcal{P}}(X)$ ($C_{\infty\mathcal{P}}(X)$) the set of all functions $f \in C(X)$ where $Z(f)$ ($\{x : |f(x)| < \frac{1}{n}\}, \forall n \in \mathbb{N}$) contains an element of \mathcal{P} and observe that $C_K(X), C_\psi(X)(C_\infty(X))$, are special kinds of $C_{\mathcal{P}}(X)$ ($C_{\infty\mathcal{P}}(X)$). In this paper we generalize well known theorems about $C_K(X), C_\psi(X)$ and $C_\infty(X)$ for $C_{\mathcal{P}}(X)$ and $C_{\infty\mathcal{P}}(X)$. $C_{\infty\mathcal{P}}(X)$ may not be an ideal of $C(X)$. We see that $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and for each $F \in \mathcal{F}$, $X \setminus \overline{F}$ is bounded if and only if the set of non-cluster points of the filter \mathcal{F} is bounded. Consequently, if $\mathcal{P} = \{A \subsetneq_{open} X : X \setminus A \text{ is bounded}\}$ (respectively $\mathcal{P} = \{A \subsetneq X : X \setminus A \text{ is finite}\}$), $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ if and only if any union of the interior of closed bounded subsets of X is bounded (respectively the set of isolated points of X is bounded). Also we find equivalent condition for which $C_{\mathcal{P}}(X)$ and $C_{\infty\mathcal{P}}(X)$ are equal. Moreover we prove that $C_{\mathcal{P}}(X)$ is an essential (respectively free) ideal if and only if for any open set U in X there exists an open set $V \subseteq U$ such that $X \setminus V \in \mathcal{F}$ (respectively \mathcal{F} has no cluster point). Finally, we prove that $C_{\infty\mathcal{P}}(X)$ is a regular ring (respectively z -ideal) if and only if every closed \mathcal{F} - CG_δ is an open subset and a member of \mathcal{F} (respectively every cozero-set which contains a \mathcal{F} - CG_δ is an element of \mathcal{F}).

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1 Introduction

In this paper, X assumed to be a completely regular Hausdorff space. $C(X)$ ($C^*(X)$) stands for the ring of all real valued (bounded) continuous functions on X . A nonempty collection \mathcal{B} of nonempty subsets of the space X is a filter base for some filter on X if and only if the intersection of any two elements of \mathcal{B} contains an element of \mathcal{B} . If any element of \mathcal{B} is an open subset, we say \mathcal{B} is an open filter base. it is worth to mention that, in this case the filter generated by \mathcal{B} consists of all suppersets of element of \mathcal{B} . We shall say that a point $p \in X$ is a cluster point of \mathcal{F} if every neighborhood of p meets every member of \mathcal{F} , in other words, $p \in \bigcap_{F \in \mathcal{F}} \overline{F}$. Kohls in [9] has proved that the intersection of all free maximal ideals in $C^*(X)$ is precisely the set $C_\infty(X)$ consists of all continuous functions $f \in C(X)$ which vanish at infinity, in the sense that $\{x : |f(x)| \geq \frac{1}{n}\}$ is compact for each $n \in \mathbb{N}$. Kohls has also shown that the $C_K(X)$ consists of all functions in $C(X)$ with compact support is the intersection of all free ideals in $C(X)$ and all free ideals in $C^*(X)$. It is well known that $C_K(X)$ is an ideal of $C(X)$ and it is easy to see that $C_\infty(X)$ is an ideal of $C^*(X)$ but not in $C(X)$, see [5]. An ideal I of $C(X)$ is called essential ideal if $I \cap (f) \neq 0$, for every non-zero element $f \in C(X)$. In [4] it has been proved that an ideal I of $C(X)$ is an essential ideal if and only if $\bigcap Z[I]$ doesn't contain an open subset ($\text{int} \bigcap Z[I] = \emptyset$). In [3] it has been proved $C_K(X)$ is an essential ideal if and only if X is an almost locally compact (X has a dense locally compact subspace). An ideal I of $C(X)$ is called z -ideal if $Z(f) \subseteq Z(g)$, $f \in I$, then $g \in I$. In particular, maximal ideals, minimal prime ideals, and most of familiar ideals in $C(X)$ are z -ideals. see [6] and [2].

Our aim of this paper is to reveal some important properties of a special kind of generalized form of $C_K(X)$ and $C_\infty(X)$, which denoted by $C_{\mathcal{P}}(X)$ and $C_{\infty\mathcal{P}}(X)$. In section 2, some examples of those subrings are given and it is shown that $C_{\mathcal{P}}(X)$ is a free ideal if and only if \mathcal{F} has no cluster point. Consequently, we observe that X is a local space (there is an open filter base \mathcal{P} for a filter \mathcal{F} which \mathcal{F} has no cluster point) if and only if there is an open filter base \mathcal{P} such that $C_{\mathcal{P}}(X)$ is a free ideal. In this section, we prove $C_{\mathcal{P}}(X)(C_{\infty\mathcal{P}}(X))$ is a zero ideal if and only if any element of \mathcal{P} (\mathcal{F}) is dense in X . A subset A of X is called \mathcal{F} -CG $_\delta$, if $A = \bigcap_{i=1}^\infty A_i$, where each A_i is an open subset, $X \setminus A_i, \overline{A_{i+1}}$ are completely separated and $A_i \in \mathcal{F}$. We also show that $C_{\mathcal{P}}(X) = C_{\infty\mathcal{P}}(X)$ if and only if every closed \mathcal{F} -CG $_\delta$ is an element of \mathcal{F} . We give an example of an open filter base \mathcal{P} over which $C_{\infty\mathcal{P}}(X)$ is not an ideal of $C(X)$. It is also shown that $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and for any $F \in \mathcal{F}$, $X \setminus \overline{F}$ is bounded if and only if the set of non-cluster points of the filter \mathcal{F} is bounded which generalizes [5, Theorem 1.3]. Consequently, If X is a pseudocompact, then for any open filter base \mathcal{P} , $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$.

Section 3 is devoted to essentiality of $C_{\mathcal{P}}(X)$ and ideals in $C_{\infty\mathcal{P}}(X)$. An ideal E in $C_{\infty\mathcal{P}}(X)$ is an essential ideal if and only if $\bigcap Z[E]$ doesn't contain $X \setminus F$ for any non-dense subset $F \in \mathcal{F}$. we observe that $C_{\mathcal{P}}(X)$ is an essential ideal if and only if for any open set U in X there is an open subset $V \subseteq U$ such that $X \setminus V \in \mathcal{F}$ (for any non-empty closed subset G in X there is a closed subset $F \supseteq G$ in \mathcal{F}).

In section 4, we have proved that $C_{\infty\mathcal{P}}(X)$ is a z -ideal if and only if every cozero-set which contains a \mathcal{F} - CG_δ is an element of \mathcal{F} . Also we have proved $C_{\infty\mathcal{P}}(X)$ is a regular ring(in the sense of von Neumann) if and only if every closed \mathcal{F} - CG_δ is an open subset and belongs to \mathcal{F} . Finally, we see that if \mathcal{P} is equal the set of all open subsets of a non-Lindelöf space X whose complements are Lindelöf, then \mathcal{P} is an open filter and $C_{\infty\mathcal{P}}(X)$ is a regular ring if and only if every closed \mathcal{P} - CG_δ is an open subset.

2 $C_{\mathcal{P}}(X)$ and $C_{\infty\mathcal{P}}(X)$

Definition 2.1. Let \mathcal{P} be a base for a filter on topological space X , in which any element is an open subset, then we define the family $C_{\mathcal{P}}(X)$ to be the set of all functions f in $C(X)$ for which $Z(f)$ contains an element of \mathcal{P} . And $C_{\infty\mathcal{P}}(X)$ denotes the family of all functions $f \in C(X)$ for which the set $\{x : |f(x)| < \frac{1}{n}\}$ contains an element of \mathcal{P} , for all $n \in \mathbb{N}$.

From now on all, assume that \mathcal{P} is an open filter base for a filter \mathcal{F} .

Lemma 2.2. (a). $C_{\mathcal{P}}(X)$ is a z -ideal of $C(X)$ contained in $C_{\infty\mathcal{P}}(X)$.

(b). $C_{\mathcal{P}}(X) = \sum_{A \in \mathcal{P}} O_A = \bigcup_{A \in \mathcal{P}} O_A$

(c). $C_{\infty\mathcal{P}}(X)$ is a proper subring of $C(X)$.

Proof.(a). By definition of $C_{\mathcal{P}}(X)$ and as such that \mathcal{P} is a base filter, it is easily seen that $C_{\mathcal{P}}(X)$ is a z -ideal and contained in $C_{\infty\mathcal{P}}(X)$.

(b). let $f \in C_{\mathcal{P}}(X)$. Then there exist $A \in \mathcal{P}$ such that $A \subseteq Z(f)$. Hence $A \subseteq \text{int}Z(f)$, i.e, $f \in O_A \subseteq \sum_{A \in \mathcal{P}} O_A$. If $f \in \sum_{A \in \mathcal{P}} O_A$, then there exist $f_1 \in O_{A_1}, \dots, f_n \in O_{A_n}$ such that $f = f_1 + \dots + f_n$, then $\bigcap_{i=1}^{i=n} A_i \subseteq \bigcap_{i=1}^{i=n} \text{int}Z(f_i) \subseteq Z(f)$. But $\bigcap_{i=1}^{i=n} A_i \in \mathcal{P}$ so $f \in C_{\mathcal{P}}(X)$. The proof of the second equality is trivial.

(c). First we observe that $C_{\infty\mathcal{P}}(X)$ is a proper subset of $C(X)$, for if $C_{\infty\mathcal{P}}(X) = C(X)$, then $\emptyset \in \mathcal{P}$, which is a contradiction. On the other hand we have $\{x : |f(x) - g(x)| < \frac{1}{n}\} \supseteq \{x : |f(x)| < \frac{1}{2n}\} \cap \{x : |g(x)| < \frac{1}{2n}\}$ and $\{x : |f(x)g(x)| < \frac{1}{n}\} \supseteq \{x : |f(x)| < \frac{1}{\sqrt{n}}\} \cap \{x : |g(x)| < \frac{1}{\sqrt{n}}\}$. So by definition, $C_{\infty\mathcal{P}}(X)$ is a proper subring of $C(X)$.

Example 2.3. Let X be a non-compact Hausdorff space X and $\mathcal{P} = \{A \subsetneq X : X \setminus A \text{ is compact}\}$. Then \mathcal{P} is an open filter base, $C_{\mathcal{P}}(X) = C_K(X)$ and

$C_{\infty\mathcal{P}}(X) = C_{\infty}(\overline{X})$. For, let $f \in C_{\mathcal{P}}(X)$. Hence there exist $A \in \mathcal{P}$ such that $A \subseteq Z(f)$, i.e., $\overline{X \setminus Z(f)} \subseteq X \setminus A$, but $X \setminus A$ is compact hence $f \in C_K(X)$. If $f \in C_K(X)$, then $\overline{X \setminus Z(f)}$ is compact and $X \setminus \overline{X \setminus Z(f)} \subseteq Z(f)$, i.e., $f \in C_{\mathcal{P}}(X)$. Similarly we may prove that $C_{\infty\mathcal{P}}(X) = C_{\infty}(X)$.

Note that if X is a compact Hausdorff space, then for any open filter base \mathcal{P} , we have $C_{\mathcal{P}}(X) \subseteq C_{\infty\mathcal{P}}(X) \subsetneq C_{\infty}(X) = C(X)$. Because $1 \in C(X)$ but $1 \notin C_{\infty\mathcal{P}}(X)$.

Example 2.4. Let \mathcal{P} be the family of all open subsets of a non-Lindelöf space X whose complements are Lindelöf. Then \mathcal{P} is an open filter and

$$C_{\infty\mathcal{P}}(X) = \{f : \overline{X \setminus Z(f)} \text{ is a Lindelöf subset of } X\},$$

$$C_{\mathcal{P}}(X) = \{f : \overline{X \setminus Z(f)} \text{ is a Lindelöf subset of } X\}.$$

For let $f \in C_{\infty\mathcal{P}}(X)$, then $\{x : |f(x)| \geq \frac{1}{n}\} \subseteq X \setminus A$ for some $A \in \mathcal{P}$, is a Lindelöf subset of X . On the other hand $X \setminus Z(f) = \bigcup_{n=1}^{\infty} \{x : |f(x)| \geq \frac{1}{n}\}$, hence $X \setminus Z(f)$ is a Lindelöf subset of X , by Theorem 3.8.5 in [7]. If $X \setminus Z(f)$ be a Lindelöf subset of X , then $\{x : |f(x)| \geq \frac{1}{n}\}$ is a Lindelöf subset of X so $\{x : |f(x)| < \frac{1}{n}\}$ contains an element of \mathcal{P} , i.e., $f \in C_{\infty\mathcal{P}}(X)$. Similarly we may prove that $C_{\mathcal{P}}(X) = \{f : \overline{X \setminus Z(f)} \text{ is a Lindelöf subset of } X\}$.

Proposition 2.5. If the complement of any element of \mathcal{P} is Lindelöf, then $C_{\infty\mathcal{P}}(X) \subseteq \bigcap_{p \in \nu X \setminus X} M^p$. Where by νX we mean the real compactification of X .

Proof. Let $f \in C_{\infty\mathcal{P}}(X)$ and M^p be a free real maximal ideal. For any $x \in X \setminus Z(f)$ there exist $f_x \in M^p$ such that $x \in X \setminus Z(f_x)$. Hence $X \setminus Z(f) \subseteq X \setminus Z(g)$ for some $g \in M^p$, because M^p is real and $X \setminus Z(g)$ is Lindelöf. Therefore $Z(g) \subseteq Z(f)$. But M^p is a z -ideal so $f \in M^p$, i.e., $C_{\infty\mathcal{P}}(X) \subseteq \bigcap_{p \in \nu X \setminus X} M^p$.

Recall that, a subset A of X is called bounded(relative pseudocompact) subset, if for every function $f \in C(X)$, $f(A)$ is a bounded subset of \mathbb{R} .

Example 2.6. Let \mathcal{P} be equal to the set of all open subsets of non-pseudocompact space X which complements are bounded, then \mathcal{P} is an open filter base and

$$C_{\mathcal{P}}(X) = C_{\psi}(X) = \{f : \overline{X \setminus Z(f)} \text{ is pseudocompact}\},$$

$$C_{\infty\mathcal{P}}(X) = \{f : \{x : |f(x)| \geq \frac{1}{n}\} \text{ is pseudocompact}\}.$$

Suppose that $f \in C_{\mathcal{P}}(X)$, then $\overline{X \setminus Z(f)} \supseteq A$ for some $A \in \mathcal{P}$. Hence $\overline{X \setminus Z(f)} \subseteq X \setminus A$ whose implies that $\overline{X \setminus Z(f)}$ is a bounded subset. And now by [10, Theorem 2.1], $\overline{X \setminus Z(f)}$ is pseudocompact, i.e., $f \in C_{\psi}(X)$. If $f \in C_{\psi}(X)$, then $X \setminus \overline{X \setminus Z(f)}$ is an element of \mathcal{P} and $Z(f) \supseteq X \setminus \overline{X \setminus Z(f)}$, i.e., $f \in C_{\mathcal{P}}(X)$. About $C_{\psi}(X)$, you can see [10].

Remark 2.7. If \mathcal{P} is equal to the set of all subsets of X which complements are finite, then $C_{\mathcal{P}}(X) = C_F(X)$, and $C_{\infty\mathcal{P}}(X) = \{f : \{x : |f(x)| \geq \frac{1}{n}\} \text{ is finite}\}$. In this case $C_{\infty\mathcal{P}}(X) = C_F(X)$ if and only if the set of isolated points of X is finite. Let $\{x_1, x_2, \dots, x_n, \dots\}$ be a subset of isolated points in

X . Define $f_n(x) = \begin{cases} \frac{1}{n} & x = n \\ 0 & x \neq n \end{cases}$ and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. Then $f \in C(X)$ and $f(x_n) = \frac{1}{n}$, $X \setminus Z(f) = \{x_1, x_2, \dots\}$, i.e, $f \notin C_F(X)$. On the other hand $\{x : |f(x)| < \frac{1}{n}\} \supseteq X \setminus \{x_1, x_2, \dots, x_n\}$ so $f \in C_{\infty\mathcal{P}}(X)$. This contradicts hypothesis. Conversely, let $f \in C_{\infty\mathcal{P}}(X)$ and $f \notin C_F(X)$. Then there is $\{x_1, x_2, \dots, x_n, \dots\}$ such that $f(x_n) \neq 0$, so $|f(x_n)| > \frac{1}{k_n}$ for some $k_n \in \mathbb{N}$, i.e, $x_n \in \{x : |f(x)| \geq \frac{1}{k_n}\}$, but $\{x : |f(x)| > \frac{1}{k_n}\}$ is a finite open set so $\{x_n\}$ is an isolated point, this is a contradiction.

Definition 2.8. A space X is called a local space provided that, there exist an open filter base \mathcal{P} for a filter \mathcal{F} on space X where \mathcal{F} has no cluster point ($\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$).

Example 2.9. Any locally compact non-compact space X is a local space. For let $\mathcal{P} = \{A \subseteq X : X \setminus A \text{ is compact}\}$, then if $x \in \bigcap_{A \in \mathcal{P}} \overline{A}$, by locally compactness of X , there exist a compact subset $K \subsetneq X$ such that $x \in \text{int}(K)$, but $X \setminus K$ is in \mathcal{P} , hence $x \in \overline{X \setminus K}$, i.e, $\text{int}(K) \cap (X \setminus K) \neq \emptyset$, this is a contradiction, thus $\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$.

Example 2.10. Let X be an uncountable space in which all points are isolated points except for a distinguished point s . A neighborhood of s is any set containing s which complement is countable, so any set contains s is closed. X is a local space, for let $Y = \{x_1, x_2, \dots\}$ being a countable subset of X , where $s \notin Y$. Set $A_n = \{x_n, x_{n+1}, \dots\}$, which is a subset of Y . Let $\mathcal{P} = \{A_n : n \in \mathbb{N}\}$, then \mathcal{P} is an open filter base on X , for any element of \mathcal{P} is a non-empty open subset and the intersection of any two element of \mathcal{P} is an element of \mathcal{P} . Now $X \setminus A_n$ for any $n \in \mathbb{N}$ is a neighborhood of s so $s \notin \bigcap_{A_n \in \mathcal{P}} \overline{A_n}$, thus $\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$. see [8, 4. M].

In the following example we observe that there is a basically disconnected space which is not a local space.

Example 2.11. For each $n \in \mathbb{N}$ let $A_n = \{n, n+1, \dots\}$ and $E = \{A_n : n \in \mathbb{N}\}$, then E is a base for a free ultrafilter say \mathcal{E} on \mathbb{N} . Let $X = \mathbb{N} \cup \{\sigma\}$ which points in \mathbb{N} are isolated point and a neighborhood of σ is of the form $U \cup \{\sigma\}$ which $U \in \mathcal{E}$. Note that any set contains σ is closed. Now if there is an open base \mathcal{P} for some filter \mathcal{F} on X such that \mathcal{F} has no cluster point, then there exist $F \in \mathcal{F}$ such that $\sigma \notin \overline{F}$, but σ has a neighborhood say $U \cup \{\sigma\}$ such that $U \in \mathcal{E}$ and $U \cup \{\sigma\} \subseteq X \setminus \overline{F}$. as such as E is a base for \mathcal{E} , there exist $n \in \mathbb{N}$ such that $A_n = \{n, n+1, \dots\} \cup \{\sigma\} \subseteq U \cup \{\sigma\} \subseteq X \setminus \overline{F}$. On the other hands for points $x = 1, 2, \dots, n-1$ there exist F_1, F_2, \dots, F_{n-1} in \mathcal{F} , such that $i \in X \setminus F_i$ for $1 \leq i \leq n$ so $X \subseteq (X \setminus \overline{F}) \cup \bigcap_{i=1}^{n-1} (X \setminus F_i)$, therefore $\emptyset \in \mathcal{F}$. This a contradiction, i.e, X is not a local space. see [8, 4. N].

We have already observe that $C_{\mathcal{P}}(X)$ is a z -ideal in $C(X)$. In the following proposition we find a condition over which $C_{\mathcal{P}}(X)$ is a free ideal.

Proposition 2.12. Let \mathcal{P} be an open filter base for filter \mathcal{F} . Then $C_{\mathcal{P}}(X)$ is a free ideal if and only if \mathcal{F} has no cluster point ($\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$).

Proof. Let $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$. Then there exist $x \in \bigcap_{F \in \mathcal{F}} \overline{F}$. By hypothesis, there is $f \in C_{\mathcal{P}}(X)$ such that $x \in X \setminus Z(f)$. On the other hand there is $A \in \mathcal{F}$ such that $X \setminus Z(f) \subseteq X \setminus A$. Hence $(X \setminus Z(f)) \cap A = \emptyset$. But $x \in \overline{A}$ implies that $(X \setminus Z(f)) \cap A \neq \emptyset$. This is a contradiction. Now let $x \in X$. We have $\bigcap_{F \in \mathcal{F}} \overline{F} = \emptyset$, so there exist $F \in \mathcal{P}$ such that $x \notin \overline{F}$. By completely regularity of X , there exist $f \in C(X)$ such that $f(x) = 1$, $f(\overline{F}) = 0$. Hence $f \in C_{\mathcal{P}}(X)$ and $x \notin Z(f)$, i.e, $C_{\mathcal{P}}(X)$ is a free ideal.

it is easily seen that X is a locally compact non-compact if and only if $\mathcal{P} = \{A \subseteq X : X \setminus A \text{ is compact}\}$ is an open filter base with no cluster point. So by the above proposition we have the following corollary.

Corollary 2.13. $C_K(X)$ is a free ideal if and only if X is a locally compact non-compact space.

Corollary 2.14. A space X is a local space if and only if $C_{\mathcal{P}}(X)$ is a free ideal for some open filter base \mathcal{P} on X .

Proof. By 2.12, the verification is immediate.

Naturally, there is a question which $C_{\mathcal{P}}(X)$ or even $C_{\infty\mathcal{P}}(X)$ can be zero ideal? In the following proposition we answer this question.

Proposition 2.15. Let \mathcal{P} be an open filter base. The following statement are equivalent.

- (a). Every element of \mathcal{P} is dense in X .
- (b). $C_{\infty\mathcal{P}}(X) = (0)$
- (c). $C_{\mathcal{P}}(X) = (0)$

Proof. (a) \Rightarrow (b). Let for every $A \in \mathcal{P}$, $\overline{A} = X$ and $f \in C_{\infty\mathcal{P}}(X)$, then the set $\{x : |f| \leq \frac{1}{n}\} = X$ so for any $n > 1$ we have $\{x : |f| < \frac{1}{n-1}\} \supseteq \{x : |f| \leq \frac{1}{n}\} = X$, i.e, $f = 0$.

(b) \Rightarrow (c). This is evident, for $C_{\mathcal{P}}(X) \subseteq C_{\infty\mathcal{P}}(X)$.

(c) \Rightarrow (a). Suppose that $C_{\mathcal{P}}(X) = 0$ and $A \in \mathcal{P}$. If $\overline{A} \neq X$, then there exist $x \in X \setminus \overline{A}$, hence we define $f \in C(X)$ such that $f(x) = 1$, $f(\overline{A}) = 0$, i.e, $f \in C_{\mathcal{P}}(X) = 0$, which implies that $f = 0$, this is a contradiction.

Corollary 2.16. Let $X = \mathbb{Q}$ and $\mathcal{P} = \{A \subset \mathbb{Q} : \mathbb{Q} \setminus A \text{ is compact}\}$, then $C_{\infty\mathcal{P}}(X) = C_{\infty}(X) = (0)$.

Proof. Every element of \mathcal{P} is dense in X , so by above proposition, $C_{\infty\mathcal{P}}(X) = C_{\infty}(X) = (0)$.

Definition 2.17. Let \mathcal{F} be a filter on X , A is subset of X where $A = \bigcap_{i=1}^{\infty} A_i$, each A_i is an open subset, $X \setminus A_i$, $\overline{A_{i+1}}$ are completely separated and $A_i \in \mathcal{F}$, we say A is a \mathcal{F} - CG_δ set.

Example 2.18. Let $\mathcal{F} = \{F \subsetneq X : X \setminus F \text{ is compact}\}$ and X is a non-compact space. Then the complement of every open locally compact σ -compact subspace A is a \mathcal{F} - CG_δ . By [7, 3. p. 250], $A = \bigcup_{i=1}^{i=\infty} A_i$ which $A_i \subseteq \text{int} A_{i+1}$ and each A_i is compact so $X \setminus A$ is a \mathcal{F} - CG_δ .

In the following lemma we characterize a closed \mathcal{F} - CG_δ set.

Lemma 2.19. Let A be a closed subset of space X . Then A is a \mathcal{F} - CG_δ set if and only if $A = Z(f)$ for some $f \in C_{\infty\mathcal{P}}(X)$.

Proof. Let A be a \mathcal{F} - CG_δ , so $A = \bigcap_{n=1}^{n=\infty} A_n$ such that for each $n \in \mathbb{N}$, A_n is an element of \mathcal{F} and $X \setminus A_n$, $\overline{A_{n+1}}$ are completely separated. Now for each $n \in \mathbb{N}$, there exist $f_n \in C(X)$ such that $f_n(\overline{A_{n+1}}) = 0$, $f_n(X \setminus A_n) = 1$, then $f = \sum \frac{1}{2^n} f_n$ is an element of $C(X)$, by Weierstrass M-test. Clearly $A = Z(f)$. We claim that $f \in C_{\infty\mathcal{P}}(X)$. Let $x_0 \in A_{n+1}$, then $f_1(x_0) = f_2(x_0) = \dots f_n(x_0) = 0$ and so $f(x_0) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \dots \leq \frac{1}{2^n} < \frac{1}{n}$. Therefore $x_0 \in \{x : |f(x)| < \frac{1}{n}\}$, and hence $A_{n+1} \subseteq \{x : |f(x)| < \frac{1}{n}\}$, i.e, $f \in C_{\infty\mathcal{P}}(X)$. Suppose that $A = Z(f)$ for some $f \in C_{\infty\mathcal{P}}(X)$. Then $A = \bigcap_{n=1}^{\infty} \{x : |f(x)| < \frac{1}{n}\}$, $A_n = \{x : |f(x)| < \frac{1}{n}\} \in \mathcal{F}$ for each $n \in \mathbb{N}$, $X \setminus A_n$ and $\overline{A_{n+1}}$ are disjoint zero-sets, and hence completely separated, i.e, A is a \mathcal{F} - CG_δ .

We know that $C_{\mathcal{P}}(X)$ is a subset of $C_{\infty\mathcal{P}}(X)$. In the following proposition we find filter \mathcal{F} for which $C_{\mathcal{P}}(X)$ and $C_{\infty\mathcal{P}}(X)$ are equal.

Proposition 2.20. $C_{\infty\mathcal{P}}(X) = C_{\mathcal{P}}(X)$ if and only if every closed \mathcal{F} - CG_δ is an element of \mathcal{F} .

Proof. Suppose that condition holds. It is enough to prove that $C_{\infty\mathcal{P}}(X) \subseteq C_{\mathcal{P}}(X)$. Let $f \in C_{\infty\mathcal{P}}(X)$, then by Lemma 2.12, $Z(f)$ is a closed \mathcal{F} - CG_δ . Hence $Z(f)$ contains an element of \mathcal{P} , i.e, $f \in C_{\mathcal{P}}(X)$. Conversely, suppose that $C_{\infty\mathcal{P}}(X) = C_{\mathcal{P}}(X)$ and A is a closed \mathcal{F} - CG_δ . By lemma 2.12, $A = Z(f)$ for some $f \in C_{\infty\mathcal{P}}(X)$, now $f \in C_{\mathcal{P}}(X)$ implies that $A = Z(f)$ contains an element of \mathcal{P} , i.e, $A \in \mathcal{F}$.

In the above proposition we seen that if every closed \mathcal{F} - CG_δ is an element of \mathcal{F} , then $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$. But in general, $C_{\infty\mathcal{P}}(X)$ may not be an ideal of $C(X)$ as we will see in the sequel.

Example 2.21. Let $\mathcal{P} = \{\mathbb{R} \setminus [\frac{1}{n}, n] : n \in \mathbb{N}\}$, then it is easily seen that \mathcal{P} is an open filter base on \mathbb{R} , and we show that $C_{\infty\mathcal{P}}(\mathbb{R}) \subsetneq C_\infty(\mathbb{R})$ is not an ideal

of $C(\mathbb{R})$. For, Let $f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ \frac{1}{x^2} & 1 \leq x \end{cases}$, $g(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ x^2 & 1 \leq x \end{cases}$, then

$f \in C_{\infty\mathcal{P}}(\mathbb{R})$, $g \in C(\mathbb{R})$ and we have $(fg)(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$, is not in $C_{\infty\mathcal{P}}(\mathbb{R})$, for $\{x : |(fg)(x)| < 1\} = (-\infty, 1)$. Since $\frac{1}{x^2+1} \in C_{\infty}(\mathbb{R})$ but is not in $C_{\infty\mathcal{P}}(\mathbb{R})$, so $C_{\infty\mathcal{P}}(\mathbb{R}) \subsetneq C_{\infty}(\mathbb{R})$.

In the following, we give an example of open filter base \mathcal{P} for which $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$.

Example 2.22. If \mathcal{P} is equal to the set of all open subsets which complements are Lindelöf, then we know that $C_{\infty\mathcal{P}}(X) = \{f : X \setminus Z(f) \text{ is Lindelöf subset of } X\}$, in this case $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$. For let $f, g \in C_{\infty\mathcal{P}}(X)$, then $X \setminus Z(f+g) \subseteq (X \setminus Z(f)) \cup (X \setminus Z(g))$ implies $\{x : |(f+g)(x)| < \frac{1}{n}\} \supseteq X \setminus ((X \setminus Z(f)) \cup (X \setminus Z(g)))$, which contains an element of \mathcal{P} , i.e., $f+g \in C_{\infty\mathcal{P}}(X)$. If $f \in C_{\infty\mathcal{P}}(X)$ and $g \in C(X)$, then $X \setminus Z(fg) \subseteq X \setminus Z(f)$ implies that $\{x : |(f.g)(x)| < \frac{1}{n}\}$ contains an element of \mathcal{P} , i.e., $f.g \in C_{\infty\mathcal{P}}(X)$.

F. Azarpanah and T. Soundarajan in [5] have found some equivalent conditions for which $C_{\infty}(X)$ is an ideal of $C(X)$ (e.g., $\mathcal{P} = \{A \subsetneq X : X \setminus A \text{ is compact}\}$). In the following theorem we find some equivalent conditions for a larger class of \mathcal{P} which $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and by this theorem give several corollary.

Theorem 2.23. The following statements are equivalent.

- (a). $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and for any $F \in \mathcal{F}$, $X \setminus \overline{F}$ is bounded.
- (b). The set of non-cluster points of filter \mathcal{F} is bounded.
- (c). The complement of every closed \mathcal{F} - CG_{δ} is bounded.

Proof. (a) \Rightarrow (b). Let A be the set of non-cluster points of filter \mathcal{F} . If A is not bounded then there exist $h \in C(X)$ and discrete subset $C = \{x_1, x_2, x_3, \dots\} \subseteq A$ such that for each $n \in \mathbb{N}$, $h(x_n) \geq n$. A is open so any $\{x_n\}$ is an isolated point, therefore we can define $f_n(x) = \begin{cases} \frac{1}{n} & x = x_n \\ 0 & x \neq x_n \end{cases}$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ such that $f_n \in C(X)$ and so $f \in C(X)$. We have $\{x : |f| < \frac{1}{n}\} = \{x_{n+1}, x_{n+2}, \dots\}$. Now any $x_n \in X \setminus \overline{F_n}$ for some $F_n \in \mathcal{F}$. This implies that $X \setminus \{x_1, \dots, x_n\}$ contains an element of \mathcal{F} so of \mathcal{P} , i.e., $f \in C_{\infty\mathcal{P}}(X)$. But we have $\{x : |fh| < \frac{1}{n}\} = X \setminus \{x_1, x_2, \dots\}$, if there is $P \in \mathcal{P}$ such that $X \setminus \{x_1, x_2, \dots\} \supseteq P$, then $\{x_1, x_2, \dots\} \subseteq X \setminus \overline{P}$, which contradict by hypothesis, so $fh \notin C_{\infty\mathcal{P}}(X)$, i.e., $C_{\infty\mathcal{P}}(X)$ is not an ideal of $C(X)$, this is a contradiction.

(b) \Rightarrow (c). easily seen that the complement of every closed \mathcal{F} - CG_{δ} is a subset of non-cluster points of filter \mathcal{F} so is bounded.

(c) \Rightarrow (a). By Lemma 2.2, $C_{\infty\mathcal{P}}(X)$ is a subring of $C(X)$, it is enough to prove that $fg \in C_{\infty\mathcal{P}}(X)$ for every $f \in C(X)$ and $g \in C_{\infty\mathcal{P}}(X)$. By Lemma 2.12, $Z(g)$ is a \mathcal{F} - CG_{δ} and hence, by (c), $Y = X \setminus Z(g)$ is a bounded subset of X , so $f(Y)$ is a bounded subset of R . Now it is easy $g^{\frac{1}{3}} \in C_{\infty\mathcal{P}}(X)$,

since $g \in C_{\infty\mathcal{P}}(X)$, moreover $(f(g)^{\frac{1}{3}})(X) = (f(g)^{\frac{1}{3}})(Y) \cup \{0\}$. Since $f(Y)$ is a bounded subset of \mathbb{R} and $g^{\frac{1}{3}} \in C_{\infty\mathcal{P}}(X)$ is a bounded function on X ($X \setminus Z(g)$ is bounded), we get $(f(g)^{\frac{1}{3}})(Y)$ is a bounded set in R implies that $(f(g)^{\frac{1}{3}})$ is a bonded on X and so belong to $C^*(X)$. Since $C_{\infty\mathcal{P}}(X)$ is a ring, $g^{\frac{2}{3}} \in C_{\infty\mathcal{P}}(X)$. However, $C_{\infty\mathcal{P}}(X)$ is an ideal of $C^*(X)$. Therefore $fg = (f(g)^{\frac{1}{3}})(g^{\frac{2}{3}}) \in C_{\infty\mathcal{P}}(X)$, thus $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$. Now let $F \in \mathcal{F}$ and $X \setminus \overline{F}$ is unbounded, so contains infinity set of isolated points say $A = \{x_1, x_2, \dots\} \subseteq (X \setminus \overline{F})$, such that A is unbounded. We claim that $X \setminus A$ is a \mathcal{F} -CG $_\delta$. For $X \setminus A = \bigcap_{i=1}^\infty B_i$ which $B_i = X \setminus \{x_1, \dots, x_i\}$, $B_i \in \mathcal{F}$ and $X \setminus B_i, \overline{B_{i+1}}$ are completely separated so by hypothesis, $X \setminus (X \setminus A) = A$ is bounded, this is a contradiction.

Corollary 2.24. If X is a pseudocompact space, then for any open filter base \mathcal{P} , $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$.

Proof. If X is a completely regular pseudocompact Hausdorff space and \mathcal{P} be an open filter base for filter \mathcal{F} , then any subset of X is bounded so the set of non-cluster point of \mathcal{F} is bounded, thus by theorem 2.23, $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$.

Corollary 2.25. Let X be a local space. Then for any open filter base \mathcal{P} , $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and for any $A \in \mathcal{P}$, $X \setminus \overline{A}$ is bounded if and only if X is a pseudocompact non-compact space.

Proof. If X is pseudocompact, then by corollary 2.24, for any open filter base \mathcal{P} , $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ and for any $A \in \mathcal{P}$, $X \setminus \overline{A}$ is bounded. Now let X be a local space, then there exist an open filter base \mathcal{P} for some filter \mathcal{F} on X such that \mathcal{F} has no cluster point so X is the set of non-cluster point of filter \mathcal{F} . Hence by 2.23, X is bounded, i.e, X is pseudocompact.

Corollary 2.26. Let X be a non-pseudocompact space and $\mathcal{P} = \{A \subsetneq_{\text{open}} X : X \setminus A \text{ is bounded}\}$. Then $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ if and only if any union of the interior of closed bounded subsets is a bounded subset.

Proof. If any union of the interior of closed bounded subsets is a bounded subset, then the set of non-cluster point of open filter \mathcal{P} is bounded so by theorem 2.23, $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$. Conversely, If $A = \bigcup_{\alpha \in S} \text{int} A_\alpha$ where for each $\alpha \in S$, A_α is a closed bounded set, then we have $X \setminus A_\alpha \in \mathcal{P}$ and $\text{int} A_\alpha = X \setminus \overline{(X \setminus A_\alpha)}$, so A is contained in the set of non-cluster points of open filter \mathcal{P} , i.e, A is bounded.

Corollary 2.27. Let X be an infinite space and $\mathcal{P} = \{A \subsetneq X : X \setminus A \text{ is finite}\}$. Then $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$ if and only if the set of isolated points of X is bounded.

Proof. Let $A \in \mathcal{P}$ be an open subset. We have $X \setminus \overline{A}$ is an open finite subset, thus the set of non-cluster point of \mathcal{P} is contained in the set of isolated points of X , so is bounded, hence by theorem 2.23, $C_{\infty\mathcal{P}}(X)$ is an ideal of $C(X)$. Conversely, let A be the set of isolated points of X . Then $A = \bigcup_{x \in A} \{x\}$, each $\{x\}$ is a clopen subset, $X \setminus \{x\} \in \mathcal{P}$ and $\{x\} = \text{int}\{x\} = X \setminus \overline{(X \setminus \{x\})}$, so A is contained in the set of non-cluster points of open filter \mathcal{P} , i.e, A is bounded.

Example 2.28. (a). If $\mathcal{P} = \{A \subsetneq \mathbb{R} : \mathbb{R} \setminus A \text{ is bounded}\}$. Then $C_{\infty\mathcal{P}}(\mathbb{R})$ is not an ideal of $C(\mathbb{R})$. Because $\bigcup_{n=1}^{\infty} (0, n)$ is not bounded.

(b). If $\mathcal{P} = \{A \subsetneq \mathbb{R} : \mathbb{R} \setminus A \text{ is finite}\}$. Then by the above corollary, $C_{\infty\mathcal{P}}(\mathbb{R})$ is an ideal of $C(\mathbb{R})$.

Remark 2.29. Any closed bounded in a normal space is a pseudocompact and any pseudocompact Lindelöf space is compact, so if X is a realcompact normal space and \mathcal{P} equal be the set of all subsets whose complements are bounded subsets of X , then $C_{\infty\mathcal{P}}(X) = C_{\infty}(X)$, for example in \mathbb{R} if \mathcal{P} equal be the set of all subsets of \mathbb{R} , whose complements are bounded, then $C_{\infty\mathcal{P}}(\mathbb{R}) = C_{\infty}(\mathbb{R})$. In particularly, if X is a Lindelöf space and \mathcal{P} equal be the set of all subsets of X , whose complements are bounded, then $C_{\infty\mathcal{P}}(X) = C_{\infty}(X)$.

3 $C_{\mathcal{P}}(X)$ as an essential ideal.

Topological space X for which $C_{\infty}(X)(C_K(X))$ is an essential ideal was characterized by F. Azarpanah, in [3]. In this section we will prove that for completely regular Hausdorff space X , $C_{\mathcal{P}}(X)$ is an essential ideal if and only if for any open set U in X there is an open set $V \subseteq U$ such that $X \setminus V \in \mathcal{F}$. We also show that an ideal E in $C_{\infty\mathcal{P}}(X)$ is an essential ideal if and only if $\bigcap Z[E]$ does not contains $X \setminus F$ for any non-dense subset $F \in \mathcal{F}$, which is a generalization of [1, Proposition, 4.6].

Proposition 3.1. An ideal E in $C_{\infty\mathcal{P}}(X)$ is an essential ideal if and only if every $F \in \mathcal{F}$ which $\bigcup_{f \in E} \text{coz}(f) \subseteq F$ is dense in X .

Proof. Let $F \in \mathcal{F}$, $\bigcup_{f \in E} \text{coz}(f) \subseteq F$ and $\overline{F} \neq X$, then there exist $x \in X$ such that $x \notin \overline{F}$, it follows that there is $f \in C(X)$ such that $f(x) = 1$, $f(F) = 0$, i.e, $f \in C_{\infty\mathcal{P}}(X)$. Now for any $g \in E$ we have $X \setminus Z(f) \subseteq X \setminus F \subseteq Z(g)$, i.e, $fg = 0$ so $(f) \cap E = 0$, which contradicts the essentiality of E . Conversely, let $0 \neq f \in C_{\infty\mathcal{P}}(X)$. So there is $a \in X$ such that $|f(a)| > \frac{1}{n}$ for some $n \in \mathbb{N}$, hence $a \in X \setminus \{x : |f| \leq \frac{1}{n}\}$, i.e, $\{x : |f| \leq \frac{1}{n}\} \neq X$, we know that $\{x : |f| < \frac{1}{n}\} \in \mathcal{F}$ so by hypothesis, $X \setminus \{x : |f| < \frac{1}{n}\} \not\subseteq \bigcap Z[E]$. Therefore there exist $b \in X \setminus \{x : |f| \leq \frac{1}{n}\}$ and $g \in E$ such that $g(b) \neq 0$, i.e, $fg \neq 0$ thus E is an essential ideal in $C_{\infty\mathcal{P}}(X)$.

Now we can prove the following theorem by adapting the proof of [3, Theorem 3.2].

Theorem 3.2. $C_{\mathcal{P}}(X)$ is an essential ideal if and only if for any open set U in X there is an open set $V \subseteq U$ such that $X \setminus V \in \mathcal{F}$.

Proof. We will prove that for every non-unit $g \in C(X)$, $C_{\mathcal{P}}(X) \cap (g) \neq 0$. Since $X \setminus Z(g)$ is an open set, then there is an open set U where $U \subseteq clU \subseteq X \setminus Z(g)$, and there is an open set $V \subseteq U$ such that $X \setminus V \in \mathcal{F}$. Then $V \subseteq U \subseteq X \setminus Z(g)$. Define $f \in C(X)$ such that $f(X \setminus V) = 0, f(x) = 1$ for some $x \in V$. Since $X \setminus V \subseteq Z(f)$ so $f \in C_{\mathcal{P}}(X)$. At the other hand $Z(g) \subseteq X \setminus V \subseteq Z(f)$ so $fg \neq 0$ and $fg \in C_{\mathcal{P}}(X) \cap (g)$.

conversely, Let U be a proper open set in X . By regularity of X , there exist a non-empty open set V such that $V \subseteq clV \subseteq U$. Now find $f \in C(X)$ where $f(clV) = \{1\}, f(x) = 0$ for some $x \notin U$. If $X \setminus V \in \mathcal{F}$, there is nothing to prove. Suppose $X \setminus V \notin \mathcal{F}$. If $V \subseteq Z(h)$ for every $h \in C_{\mathcal{P}}(X)$, then $V \subseteq \bigcap Z[C_{\mathcal{P}}(X)]$, which implies that $C_{\mathcal{P}}(X)$ is not an essential ideal, by [4, Theorem 3.1]. Therefore there is some $h \in C_{\mathcal{P}}(X)$ such that $V \cap (X \setminus Z(h)) \neq \emptyset$, i.e., there is some $x_0 \in V$ for which $h(x_0) \neq 0$. Clearly $fh \in C_{\mathcal{P}}(X)$. So $W = X \setminus Z(fh)$ is contained in $X \setminus F$ for some $F \in \mathcal{F}$. If $W' = W \cap V$, then W' is a non-empty open set in U and $X \setminus W' \in \mathcal{F}$.

Corollary 3.3. If \mathcal{P} is equal to the family of all open subsets of non-Lindelöf space X which complements are Lindelöf, then $C_{\infty\mathcal{P}}(X)$ ($C_{\mathcal{P}}(X)$) is an essential ideal if and only if for any open set U in X there is an open Lindelöf subset $V \subseteq U$.

Proof. By Theorem 3.2, this is evident.

4 $C_{\infty\mathcal{P}}(X)$ as a z -ideal and a regular ring.

We know that $C_{\infty\mathcal{P}}(X)$ is a subring of $C(X)$, in this section, we see that $C_{\infty\mathcal{P}}(X)$ is a z -ideal if and only if Every cozero-set which contains a \mathcal{F} - CG_δ is an element of \mathcal{F} . Also we prove that, $C_{\infty\mathcal{P}}(X)$ is a regular ring if and only if every closed \mathcal{F} - CG_δ is an open subset and belong to \mathcal{F} .

Proposition 4.1. $C_{\infty\mathcal{P}}(X)$ is a z -ideal of $C(X)$ if and only if every cozero-set which contains a \mathcal{F} - CG_δ is an element of \mathcal{F} .

Proof. Let $X \setminus Z(f)$ be a cozero-set and A be a closed CG_δ in \mathcal{F} . by Lemma 2.19, there exist $g \in C_{\infty\mathcal{P}}(X)$, such that $A = Z(g)$ so $Z(g) \subseteq X \setminus Z(f)$. Now we define $h = \frac{g^2}{f^2 + g^2}$. Then $h \in C(X)$ and $Z(g) \subseteq Z(h)$, therefore $h \in C_{\infty\mathcal{P}}(X)$. on the other hand $\{x : |h(x)| < \frac{1}{n}\} \subseteq X \setminus Z(f)$, hence $X \setminus Z(f) \in \mathcal{F}$. Conversely, first we prove that $Z(f) \subseteq Z(g), f \in C_{\infty\mathcal{P}}(X)$, implies that $g \in C_{\infty\mathcal{P}}(X)$. $Z(f) \subseteq Z(g) \subseteq \{x : |g(x)| < \frac{1}{n}\}$, for all $n \in \mathbb{N}$. But $\{x : |g(x)| < \frac{1}{n}\}$ is a cozero-set and $Z(f)$ is a closed \mathcal{F} - CG_δ so by hypothesis, $\{x : |g(x)| < \frac{1}{n}\}$ is an element of \mathcal{F} , i.e., $g \in C_{\infty\mathcal{P}}(X)$. suppose $f \in C_{\infty\mathcal{P}}(X), g \in C(X)$, then $Z(f) \subseteq Z(fg)$ show that $fg \in C_{\infty\mathcal{P}}(X)$. Thus $C_{\infty\mathcal{P}}(X)$ is a z -ideal.

Theorem 4.2. $C_{\infty\mathcal{P}}(X)$ is a regular ring if and only if every closed $\mathcal{F}\text{-}CG_\delta$ is an open subset and belongs to \mathcal{F} .

Proof. First, we prove that every closed $\mathcal{F}\text{-}CG_\delta$ is an open subset. By Lemma 2.19, every closed $\mathcal{F}\text{-}CG_\delta$ is of the form $Z(f)$ for some $f \in C_{\infty\mathcal{P}}(X)$. $Z(f) = Z(f \wedge n)$, for each $n \in \mathbb{N}$ and $\{x : |f \wedge n| < \frac{1}{n}\} = \{x : |f| < \frac{1}{n}\}$. So we can let f is bounded. Regularity of $C_{\infty\mathcal{P}}(X)$ implies that, there exist $g \in C_{\infty\mathcal{P}}(X)$ such that $f = f^2g$. Then $X \setminus Z(1 - fg) \subseteq \text{int}Z(f)$. If $x \in Z(f) \setminus \text{int}Z(f)$, then $x \in Z(1 - fg)$, which contradict $x \in Z(f)$, i.e, $Z(f)$ is an open subset. On the other hand for every $x \in X \setminus Z(f)$, $g(x) = \frac{1}{f(x)}$ and hence $g(x) \geq \frac{1}{n}$, where n is an upper bounded for $|f|$. Therefore $X \setminus Z(f) \subseteq \{x : |g| \geq \frac{1}{n}\}$, i.e, $Z(f) \supseteq \{x : |g| < \frac{1}{n}\}$. But $\{x : |g| < \frac{1}{n}\}$ contains an element of \mathcal{P} so $Z(f) \in \mathcal{F}$. Conversely, Suppose $f \in C_{\infty\mathcal{P}}(X)$. $Z(f)$ is a closed $\mathcal{F}\text{-}CG_\delta$ so by hypothesis, is an open subset which belong to \mathcal{F} . We define $g(x) = 0$ for $x \in Z(f)$ and $g(x) = \frac{1}{f(x)}$ for $x \in X \setminus Z(f)$. Then $g \in C(X)$, $f = f^2g$ and $\{x : |g| < \frac{1}{n}\} \supseteq Z(f)$. , i.e, $g \in C_{\infty\mathcal{P}}(X)$.

Corollary 4.3. (a). If \mathcal{P} equal be the family of all open subsets of non-Lindelöf space X which complements are Lindelöf, then \mathcal{P} is an open filter and $C_{\infty\mathcal{P}}(X)$ is a regular ring if and only if every closed $\mathcal{P}\text{-}CG_\delta$ is an open subset.

(b). $C_\infty(X)$ is a regular ring if and only if every open locally compact σ -compact is a compact subspace.

Proof. (a). easily seen that \mathcal{P} is an open filter. If $C_{\infty\mathcal{P}}(X)$ is a regular ring, then by Theorem 4.2, every closed $\mathcal{P}\text{-}CG_\delta$ is an open subset. Now let A be a closed $\mathcal{P}\text{-}CG_\delta$ which is an open subset. By Lemma 2.19, $A = Z(f)$ for some $f \in C_{\infty\mathcal{P}}(X)$. But $X \setminus A = X \setminus Z(f) = \bigcup_{n=1}^{\infty} \{x : |f(x)| \geq \frac{1}{n}\}$, hence by [7, Theorem 3.8.5], $X \setminus A$ is a Lindelöf subset of X , i.e, $A \in \mathcal{P}$. So by Theorem 4.2, $C_{\infty\mathcal{P}}(X)$ is a regular ring.

(b). If A is an open locally compact σ -compact subset, then $X \setminus A$ is a closed $\mathcal{P}\text{-}CG_\delta$, which $\mathcal{P} = \{A \subsetneq X : X \setminus A \text{ is compact}\}$ and $C_\infty(X) = C_{\infty\mathcal{P}}(X)$, so by Theorem 4.2, this is trivial.

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